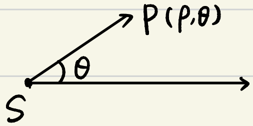
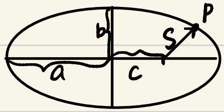
由 Kepler 三定律牛顿第二定律推导万有引力（极坐标下）

牛顿《自然哲学的数学原理》中的逻辑.



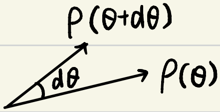
(1)



$$\rho=\frac{b^2/a}{1+\frac{c}{a}\cos\theta} \quad ①$$

令 $u(\theta)=\frac{1}{\rho}=\frac{a}{b^2}\left(1+\frac{c}{a}\cos\theta\right)$

(2)



$$\frac{dA}{dt}=\frac{\frac{1}{2}\rho\cdot\rho d\theta}{dt}=\frac{1}{2}\rho^2\dot\theta=\text{const}$$

$\Rightarrow$ 运动一周 联系① $\frac{1}{2}\rho^2\dot\theta\cdot T=\pi ab$ ②

(3) $\frac{T^2}{a^3}=\text{const}$ ③

(4) 极坐标系下有心力作用下的牛顿第二定律

$$\begin{cases} m(\ddot\rho-\rho\dot\theta^2)=F \xrightarrow[\text{Binet 方程}]{u=\frac{1}{\rho},\,h=\rho^2\dot\theta} h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)=-\frac{F}{m} \quad ④ \\ m(2\dot\rho\dot\theta+\rho\ddot\theta)=0 \quad ⑤ \qquad \left(h=\rho^2\dot\theta=\frac{2\pi ab}{T}\right) \end{cases}$$

式⑤和式② 等价 $\frac{1}{\rho}\frac{d(m\rho^2\dot\theta)}{dt}=0$ 即 $m\rho^2\dot\theta=\text{const}$

将式①、②代入式④ $\frac{d^2u}{d\theta^2}+u=-\frac{c}{b^2}\cos\theta+\frac{a}{b^2}+\frac{c}{b^2}\cos\theta=\frac{a}{b^2}$

得 $\left(\frac{2\pi ab}{T}\right)^2\cdot\frac{1}{\rho^2}\cdot\frac{a}{b^2}=-\frac{F}{m}$

整理得 $F=-\frac{4\pi^2a^3m}{T^2\rho^2}=-4\pi^2m\cdot\frac{1}{\rho^2}\cdot\underbrace{\frac{a^3}{T^2}}_{\text{K3的常数}}\propto\frac{1}{\rho^2}$

极坐标系下的代数形式的万有引力定律

约化的两体问题 → 将两体问题转化为两个单体问题

解 $\vec{r_1}$、$\vec{r_2}$ ⇒ 解 $\vec{x}=\frac{m_1\vec{r_1}+m_2\vec{r_2}}{m_1+m_2}$ 和 $\vec{r}=\vec{r_1}-\vec{r_2}$ 再求解 $\vec{r_1}$、$\vec{r_2}$

(1)质心的运动　质心的位矢 $\vec{x}\equiv\frac{m_1\vec{r_1}+m_2\vec{r_2}}{m_1+m_2}$　$(m_1+m_2)\vec{x}=m_1\vec{r_1}+m_2\vec{r_2}$

∵无外力　∴ $(m_1+m_2)\frac{d^2\vec{x}}{dt^2}=\vec{0}$ ①　两体问题的质心静止或匀速运动

(2)位移向量(Displacement)的运动 质点2相对质点1的位矢 $\vec{r}=\vec{r_1}-\vec{r_2}$

(或relative position)

由原始运动方程
$$\begin{cases}\frac{d^2\vec{r_1}}{dt^2}=-\frac{Gm_2}{|\vec{r_1}-\vec{r_2}|^2}\vec{e_{21}}\\ \frac{d^2\vec{r_2}}{dt^2}=-\frac{Gm_1}{|\vec{r_1}-\vec{r_2}|^2}\vec{e_{12}}\end{cases}$$
($\vec{e_{21}}$: 由2指向1的单位矢量，即 $\hat{r}$)

相减得 $\frac{d^2\vec{r}}{dt^2}=-\frac{Gm_2}{|\vec{r}|^2}\vec{e_{21}}+\frac{Gm_1}{|\vec{r}|^2}\vec{e_{12}}=-\frac{G(m_1+m_2)}{|\vec{r}|^2}\hat{r}$

即 $\frac{m_1m_2}{m_1+m_2}\frac{d^2\vec{r}}{dt^2}=-\frac{Gm_1m_2}{|\vec{r}|^2}\hat{r}=m_1\frac{d^2\vec{r_1}}{dt^2}=m_2\frac{d^2\vec{r_2}}{dt^2}=\vec{F_{12}}$ ②

约化质量 $\mu=\frac{1}{\frac{1}{m_1}+\frac{1}{m_2}}$

两体问题的 Displacement 的运动等价约化质量 μ 在两体相互作用力下的运动

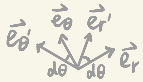
另一理解 $m\frac{d^2\vec{r}}{dt^2}=-\frac{G(m_1+m_2)}{|\vec{r}|^2}m\hat{r}$　test particle 绕 m_1+m_2 运动

在讨论行星轨道时 $m_1\gg m_2$，行星的相对位置向量 $\vec{r}$ 的运动

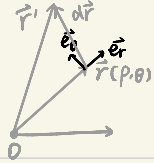
等价受到固定的恒星位置，质量为 $m_1+m_2\approx m_1$ 的质点引力作用下的运动

由①、②解出 $\vec{x}$ 和 $\vec{r}$ 的运动，代入
$$\begin{cases}\vec{r_1}=\vec{x}+\frac{m_2}{m_1+m_2}\vec{r}\\ \vec{r_2}=\vec{x}-\frac{m_1}{m_1+m_2}\vec{r}\end{cases}$$

极坐标:



$d\vec{e}_r = d\theta\,\vec{e}_\theta$
$d\vec{e}_\theta = -d\theta\,\vec{e}_r$



$d\vec{r} = \rho\, d\theta\, \vec{e}_\theta + d\rho\, \vec{e}_r$

$\vec{v} = \frac{d\vec{r}}{dt} = \underbrace{\frac{d\rho}{dt}}_{v_r}\vec{e}_r + \underbrace{\rho\frac{d\theta}{dt}}_{v_\theta}\vec{e}_\theta$

约化二体问题的 Kepler Orbit 用约化的$\vec{r}$ 极坐标系下 $\vec{r}(\rho,\theta)$ 随t的演化.

整合教材 2.1.4 & 2.1.6

(1) 由 Newton's Law 推导 机械能、角动量守恒

极坐标系下的牛二和万有引力定律

$$\begin{cases} m\left[\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2\right] = -\frac{GMm}{\rho^2} \quad ① \\ m\left(2\frac{d\rho}{dt}\frac{d\theta}{dt} + \rho\frac{d^2\theta}{dt^2}\right) = 0 \quad ② \end{cases}$$

$M = m_1 + m_2$ (两体的质量之和)

① 角动量守恒

$\vec{v} \equiv \frac{d\vec{r}}{dt}$ $\quad \vec{L} \equiv \vec{r}\times m\vec{v}$ $\quad \vec{r}\times\vec{e}_r \equiv \vec{0}$ $\quad \vec{r}\times\vec{e}_\theta = 1\cdot\vec{e}_k$

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{r}\times m\vec{v})}{dt} = m\frac{d}{dt}\left[\vec{r}\times(v_r\vec{e}_r + v_\theta\vec{e}_\theta)\right] = m\frac{d}{dt}(\rho v_\theta)\vec{e}_k = m\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right)\vec{e}_k$$

其中 $\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right) = 2\rho\frac{d\rho}{dt}\frac{d\theta}{dt} + \rho^2\frac{d^2\theta}{dt^2} \overset{式②}{=\!=} 0$ $\quad \therefore \frac{d\vec{L}}{dt} = \vec{0}$

② 机械能守恒

将 L 代入式①进行化简 $\quad L = m\rho^2\frac{d\theta}{dt} \Rightarrow m\rho\left(\frac{d\theta}{dt}\right)^2 = \frac{L^2}{m\rho^3}$

$$m\frac{d^2\rho}{dt^2} - \frac{L^2}{m\rho^3} = -\frac{GMm}{\rho^2}$$

代入 $\Phi(\rho) = -\frac{GMm}{\rho}$, $\frac{d^2\rho}{dt^2} = \frac{d\dot\rho}{dt} = \frac{d\dot\rho}{d\rho}\cdot\frac{d\rho}{dt} = \dot\rho\frac{d\dot\rho}{d\rho}$

得 $m\dot\rho\frac{d\dot\rho}{d\rho} - \frac{L^2}{m\rho^3} = -\frac{d\Phi}{d\rho}$ $\quad$ 对 ρ 积分, 得 $\frac{m}{2}\int d\dot\rho^2 - \frac{\overset{\nearrow \text{const}}{L^2}}{m}\int\frac{1}{\rho^3}d\rho = -\int d\Phi$.

即 $\frac{1}{2}m\dot\rho^2 + \frac{L^2}{2m\rho^2} + \Phi -$ 任意常数 $E_0 = 0$

机械能 $E = \frac{1}{2}mv_\rho^2 + \frac{1}{2}mv_\theta^2 + \Phi$

$$= \frac{1}{2}m\left(\frac{d\rho}{dt}\right)^2 + \frac{1}{2}m\left(\rho\frac{d\theta}{dt}\right)^2 - \frac{GMm}{\rho}$$

$$= E_0$$

(2) 由机械能、角动量守恒推导Kepler三定律.

①能量守恒推 1^{st} 即 $\rho(\theta)$ 的轨道表达式 再将 1^{st} 推广至 Kepler Orbit.

$$E=\frac{1}{2}m\dot{\rho}^2+\frac{L^2}{2m\rho^2}-\frac{GMm}{\rho} \Rightarrow \dot{\rho}=\sqrt{\frac{2E}{m}+\frac{2GM}{\rho}-\frac{L^2}{m^2\rho^2}}$$

用L消去时间,凑出轨道表达式 $\frac{d\rho}{dt}=\frac{d\rho}{d\theta}\cdot\frac{d\theta}{dt}=\frac{d\rho}{d\theta}\cdot\frac{1}{\rho}(\rho\frac{d\theta}{dt})=\frac{d\rho}{d\theta}\cdot\frac{L}{m\rho^2}$

$$\frac{d\rho}{d\theta}=\frac{m\rho^2}{L}\sqrt{\frac{2E}{m}+\frac{2GM}{\rho}-\frac{L^2}{m^2\rho^2}} \quad (书\ Eq.2.15)$$

$$=\rho^2\sqrt{\left(\frac{GMm^2}{L}\right)^2\left(1+\frac{2EL^2}{G^2M^2m^3}\right)-\left(\frac{1}{\rho}-\frac{GMm^2}{L^2}\right)^2}$$

令 $p=\frac{GMm^2}{L^2}$, $\epsilon^2=1+\frac{2EL^2}{G^2M^2m^3}$ 写作 $\frac{1}{\rho^2}\frac{d\rho}{d\theta}=\sqrt{(p\epsilon)^2-(\frac{1}{\rho}-p)^2}$

$LHS=\frac{1}{\rho^2}\frac{d\rho}{d\theta}=-\frac{d}{d\theta}(\frac{1}{\rho})$ 令 $\frac{1}{\rho}=u$ 将 u、θ 分到等式两侧

$$-\frac{du}{\sqrt{(p\epsilon)^2-(u-p)^2}}=d\theta$$

对两侧积分 $-\int\frac{du}{\sqrt{(p\epsilon)^2-(u-p)^2}}=\int d\theta$

$LHS=\arccos\frac{u-p}{p\epsilon}$ $\quad -\int\frac{dx}{\sqrt{a^2-(x+b)^2}}=\arccos\frac{x+b}{a}+C$

$\therefore\ \theta-\theta_0=\arccos\frac{u-p}{p\epsilon}$ 即 $\rho=\frac{1/p}{1+\epsilon\cos(\theta-\theta_0)}$

1° 任意常数 θ_0 与极轴的选取有关,方便讨论取 $\theta_0=0$ $\quad \rho=\frac{p}{1+\epsilon\cos\theta}$ ③

2° 仅考虑 $\epsilon>0$ 的情况, $\epsilon<0$ 可将负号以相位形式并入 θ ∴ $\epsilon=\sqrt{1+\frac{2EL^2}{GM^2m^3}}$

式③ 是圆锥曲线的表达式 $p=\frac{b^2}{a}$ $\quad \epsilon=\frac{c}{a}=\sqrt{1+\frac{2EL^2}{G^2M^2m^3}}$ 为离心率

1^{st} 提出时是以闭合(bound)轨道的观测为基础的(观测到重复才能得到科学规律)

即 $E<0$ $\ 0\leq\epsilon<1$ 行星的轨道为椭圆 而引力力心即恒星的位置在其中一个焦点上

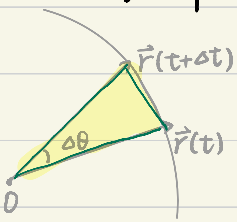
• Kepler Orbit : Bound or Unbound

1° $E<0$ $\quad 0\le\epsilon<1$ $\quad |E_k|<|E_p|$ 椭圆

2° $E=0$ $\quad \epsilon=1$ $\quad |E_k|=|E_p|$ 抛物线

3° $E>0$ $\quad \epsilon>1$ $\quad |E_k|>|E_p|$ 双曲线

② 角动量守恒推 2nd



直边三角近似

$$\Delta A=\frac{1}{2}|\vec{r}(t)|\cdot|\vec{r}(t+\Delta t)|\sin\Delta\theta$$

$$\therefore \frac{dA}{dt}=\lim_{\Delta t\to 0}\frac{\frac{1}{2}\rho(t)\rho(t+\Delta t)\sin\Delta\theta}{\Delta t}$$

$$=\frac{1}{2}\rho^2(t)\lim_{\Delta t\to 0}\frac{\sin\Delta\theta(t)}{\Delta t}$$

$$=\frac{1}{2}\rho^2(t)\frac{d\theta}{dt}$$

$$=\frac{L}{2m}=\text{const}$$

$$\frac{\sin\Delta\theta(t)}{\Delta t}=\frac{\sin\Delta\theta}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta t}$$

洛必达

$$\frac{1}{1}\cdot\frac{d\theta}{dt}$$

③ 由 1st、2nd 推 3rd

$$\rho=\frac{b^2/a}{1+\frac{c}{a}\cos\theta}\qquad\Rightarrow L=mb\sqrt{\frac{GM}{a}}$$

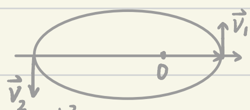
$$L=m\rho^2\frac{d\theta}{dt}=\text{const},\ E=\text{const}$$

$$\frac{dA}{dt}=\frac{L}{2m}\qquad\Rightarrow P=\frac{\pi ab\cdot 2m}{mb\sqrt{\frac{GM}{a}}}$$

$$\int_0^{2\pi}\frac{dA}{dt}d\theta=\frac{dA}{dt}\cdot P=\pi ab\qquad=\frac{2\pi}{\sqrt{GM}}\cdot a^{\frac{3}{2}}$$

整理得 $\quad \dfrac{P^2}{a^3}=\dfrac{4\pi^2}{GM}$

E、L = const 故取便于计算的近远点、



$V_\theta=|\vec{v}|\quad V_r=0$

$$E=\frac{L^2}{2m(a+c)^2}-\frac{GMm}{(a+c)}=\frac{L^2}{2m(a-c)^2}-\frac{GMm}{(a-c)}$$

$$\therefore \frac{\cancel{4}a\cancel{c}L^2}{\cancel{2}m(a^2-c^2)^{\cancel{2}}}+\frac{\cancel{2c}GMm}{\cancel{a^2-c^2}}=0$$

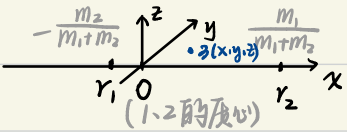
$$\therefore \frac{aL^2}{mb^2}+GMm=0$$

Hill's Problem

① 守恒量 Jacobi's Constant

m_1: 恒星　m_2: 行星　m_3: massless test particle.

$m_1 \gg m_2$　$m_1 \gg m_3$　1与2、3距离很远



3的能量、动量不一定守恒，但C_J守恒
并假设 co-rotating (共旋转)

$$C_J = \underbrace{\left(\frac{2\pi}{P}\right)^2(x^2+y^2)}_{\text{离心势能}} + \underbrace{\frac{2Gm_1}{|\vec{r}-\vec{r}_1|} + \frac{2Gm_2}{|\vec{r}-\vec{r}_2|}}_{\text{引力势能}} - \underbrace{(\dot{x}^2+\dot{y}^2+\dot{z}^2)}_{\text{动能}}$$

☆ 推论: zero-velocity surface

给定 $C_J = C$，令 $v^2 = \dot{x}^2+\dot{y}^2+\dot{z}^2 = 0$，满足 $\left(\frac{2\pi}{P}\right)^2(x^2+y^2) + \frac{2Gm_1}{|\vec{r}-\vec{r}_1|} + \frac{2Gm_2}{|\vec{r}-\vec{r}_2|} - C = v^2 = 0$ 的点构成的面

$\because v^2 \geq 0$　$\therefore$ 该曲面上势能是给定C_J的极小值 质点粒子无法越过该面.

增大C可以辅助判断拉格朗日点的位置　随C_J的增大，收敛至L_4、L_5

拉格朗日点的计算　(JWST L_2).

推导即共轨旋转下的受力平衡

偏离各个拉格朗日点后？

以L_2为例:

$$\frac{M_1}{(R+r)^2} + \frac{M_2}{r^2} \overset{r=L_2}{=} \left(\frac{M_1}{M_1+M_2}R + r\right)\frac{M_1+M_2}{R^3}$$

微扰 $r \to r+\delta r$　LHS < RHS

在此瞬间，在 test partical 的参考系下受到离心力，更远离 L_2.